Chapter 2

Limits and Continuity

Section 2.1 Rates of Change and Limits (pp. 59–69)

Quick Review 2.1

1. $f(2) = 2(2^3) - 5(2)^2 + 4 = 0$

2. $f(2) = \dfrac{4(2)^2 - 5}{2^3 + 4} = \dfrac{11}{12}$

3. $f(2) = \sin\left(\pi \cdot \dfrac{2}{2}\right) = \sin \pi = 0$

4. $f(2) = \dfrac{1}{2^2 - 1} = \dfrac{1}{3}$

5. $|x| < 4$
$-4 < x < 4$

6. $|x| < c^2$
$-c^2 < x < c^2$

7. $|x - 2| < 3$
$-3 < x - 2 < 3$
$-1 < x < 5$

8. $|x - c| < d^2$
$-d^2 < x - c < d^2$
$-d^2 + c < x < d^2 + c$

9. $\dfrac{x^2 - 3x - 18}{x + 3} = \dfrac{(x+3)(x-6)}{x+3} = x - 6,\ x \neq -3$

10. $\dfrac{2x^2 - x}{2x^2 + x - 1} = \dfrac{x(2x-1)}{(2x-1)(x+1)} = \dfrac{x}{x+1},\ x \neq \dfrac{1}{2}$

Section 2.1 Exercises

1. $\dfrac{\Delta y}{\Delta t} = \dfrac{16(3)^2 - 16(0)^2}{3 - 0} = 48$ ft/sec

2. $\dfrac{\Delta y}{\Delta t} = \dfrac{16(4)^2 - 16(0)^2}{4 - 0} = 64$ ft/sec

3. $\dfrac{\Delta y}{\Delta t} = \dfrac{16(3+h)^2 - 16(3)^2}{h}$, say $h = 0.01$

$$= \frac{16(3+0.01)^2 - 16(9)}{0.01} = \frac{16(9.0601) - 16(9)}{0.01} = \frac{144.9616 - 144}{0.01} = \frac{0.9616}{0.01} = 96.16 \text{ ft/sec}$$

Confirm Algebraically

$$\frac{\Delta y}{\Delta t} = \frac{16(3+h)^2 - 16(3)^2}{h} = \frac{16(9 + 6h + h^2) - 144}{h} = \frac{96h + 16h^2}{h} = (96 + 16h) \text{ ft/sec}$$

if $h = 0$, then $\dfrac{\Delta y}{\Delta t} = 96 \dfrac{\text{ft}}{\text{sec}}$

4. $\dfrac{\Delta y}{\Delta t} = \dfrac{16(4+h)^2 - 16(4)^2}{h}$, say $h = 0.01$

$$\frac{16(4+0.01)^2 - 16(4)^2}{0.01} = \frac{16(16.0801) - 16(16)}{0.01} = \frac{257.2816 - 256}{0.01} = \frac{1.2816}{0.01} = 128.16 \text{ ft/sec}$$

Confirm Algebraically

$$\frac{\Delta y}{\Delta t} = \frac{16(4+h)^2 - 16(4)^2}{h} = \frac{16(16 + 8h + h^2) - 256}{h} = \frac{128h + 16h^2}{h} = (128 + 16h) \text{ ft/sec}$$

if $h = 0$, then $\dfrac{\Delta y}{\Delta t} = 128 \dfrac{\text{ft}}{\text{sec}}$

5. $\lim\limits_{x \to c} (2x^3 - 3x^2 + x - 1) = 2c^3 - 3c^2 + c - 1$

6. $\lim\limits_{x \to c} \dfrac{x^4 - x^3 + 1}{x^2 + 9} = \dfrac{c^4 - c^3 + 1}{c^2 + 9}$

7. $$\lim_{x\to -1/2} 3x^2(2x-1) = 3\left(-\frac{1}{2}\right)^2\left[2\left(-\frac{1}{2}\right)-1\right]$$
$$= 3\left(\frac{1}{4}\right)(-2)$$
$$= -\frac{3}{2}$$

8. $\lim_{x\to -4} (x+3)^{1998} = (-4+3)^{1998} = (-1)^{1998} = 1$

9. $\lim_{x\to 1} (x^3+3x^2-2x-17)$
$= (1)^3+3(1)^2-2(1)-17$
$= 1+3-2-17$
$= -15$

10. $\lim_{y\to 2} \dfrac{y^2+5y+6}{y+2} = \dfrac{2^2+5(2)+6}{2+2} = \dfrac{20}{4} = 5$

11. $\lim_{y\to -3} \dfrac{y^2+4y+3}{y^2-3} = \dfrac{(-3)^2+4(-3)+3}{(-3)^2-3} = \dfrac{0}{6} = 0$

12. $\lim_{x\to 1/2} \text{int } x = \text{int}\,\dfrac{1}{2} = 0$

Note that substitution cannot always be used to find limits of the int function. Its use here can be justified by the Sandwich Theorem, using $g(x) = h(x) = 0$ on the interval (0, 1).

13. $$\lim_{x\to -2} (x-6)^{2/3} = (-2-6)^{2/3}$$
$$= \sqrt[3]{(-8)^2}$$
$$= \sqrt[3]{64}$$
$$= 4$$

14. $\lim_{x\to 2} \sqrt{x+3} = \sqrt{2+3} = \sqrt{5}$

15. (a)

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	1.566667	1.959697	1.995997	1.999600

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	2.372727	2.039703	2.003997	2.000400

The limit appears to be 2.

16. (a)

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	–1.1	–1.01	–1.001	–1.0001

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	–0.9	–0.99	–0.999	–0.9999

The limit appears to be −1.

17. **(a)**

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	–0.054402	–0.005064	–0.000827	–0.000031

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	–0.054402	–0.005064	–0.000827	–0.000031

The limit appears to be 0.

18. **(a)**

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	0.5440	0.5064	–0.8269	0.3056

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	–0.5440	–0.5064	0.8269	–0.3056

There is no clear indication of a limit.

19. **(a)**

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	2.0567	2.2763	2.2999	2.3023

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	2.5893	2.3293	2.3052	2.3029

The limit appears to be approximately 2.3.

20. **(a)**

x	–0.1	–0.01	–0.001	–0.0001
$f(x)$	0.074398	–0.009943	0.000585	0.000021

(b)

x	0.1	0.01	0.001	0.0001
$f(x)$	–0.074398	0.009943	–0.000585	–0.000021

The limit appears to be 0.

21. You cannot use substitution because the expression $\sqrt{x-2}$ is not defined at $x = -2$. Since the expression is not defined at points near $x = -2$, the limit does not exist.

22. You cannot use substitution because the expression $\frac{1}{x^2}$ is not defined at $x = 0$. Since $\frac{1}{x^2}$ becomes arbitrarily large as x approaches 0 from either side, there is no (finite) limit. (As we shall see in Section 2.2, we may write $\lim_{x\to 0} \frac{1}{x^2} = \infty$.)

23. You cannot use substitution because the expression $\dfrac{|x|}{x}$ is not defined at $x = 0$. Since $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = -1$ and $\lim\limits_{x\to 0^+} \dfrac{|x|}{x} = 1,$ the left- and right-hand limits are not equal and so the limit does not exist.

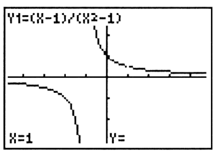
24. You cannot use substitution because the expression $\dfrac{(4+x)^2 - 16}{x}$ is not defined at $x = 0$. Since

$$\frac{(4+x)^2-16}{x} = \frac{8x+x^2}{x} = 8 + x \text{ for all } x \neq 0,$$

the limit exists and is equal to

$$\lim_{x\to 0} (8+x) = 8+0 = 8.$$

25.



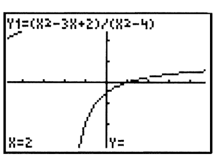
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

Algebraic confirmation:

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x+1)\,(x-1)}$$
$$= \lim_{x\to 1} \frac{1}{x+1}$$
$$= \frac{1}{1+1}$$
$$= \frac{1}{2}$$

26.



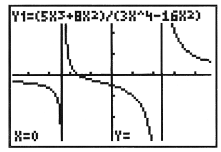
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{t\to 2} \frac{t^2-3t+2}{t^2-4} = \frac{1}{4}$$

Algebraic confirmation:

$$\lim_{t\to 2} \frac{t^2-3t+2}{t^2-4} = \lim_{t\to 2} \frac{(t-2)\,(t-1)}{(t-2)(t+2)}$$
$$= \lim_{t\to 2} \frac{t-1}{t+2}$$
$$= \frac{2-1}{2+2}$$
$$= \frac{1}{4}$$

27.



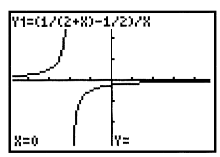
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = -\frac{1}{2}$$

Algebraic confirmation:

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \lim_{x\to 0} \frac{x^2(5x+8)}{x^2(3x^2-16)}$$
$$= \lim_{x\to 0} \frac{5x+8}{3x^2-16}$$
$$= \frac{5(0)+8}{3(0)^2-16}$$
$$= \frac{8}{-16}$$
$$= -\frac{1}{2}$$

28.

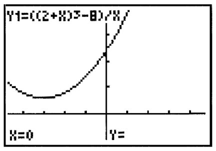


$[-4.7, 4.7]$ by $[-3.1, 3.1]$

$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = -\frac{1}{4}$$

Algebraic confirmation:

$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{2-(2+x)}{x(2+x)(2)}$$
$$= \lim_{x\to 0} \frac{-x}{x(2+x)\,(2)}$$
$$= \lim_{x\to 0} \frac{-1}{2(2+x)}$$
$$= \frac{-1}{2(2+0)}$$
$$= -\frac{1}{4}$$

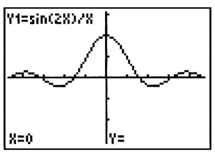
29. 

[−4.7, 4.7] by [−5, 20]

$$\lim_{x\to 0}\frac{(2+x)^3-8}{x}=12$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0}\frac{(2+x)^3-8}{x} &= \lim_{x\to 0}\frac{12x+6x^2+x^3}{x}\\ &= \lim_{x\to 0}(12+6x+x^2)\\ &= 12+6(0)+(0)^2\\ &= 12\end{aligned}$$

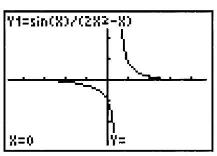
30. 

[−4.7, 4.7] by [−3.1, 3.1]

$$\lim_{x\to 0}\frac{\sin 2x}{x}=2$$

Algebraic confirmation:

$$\lim_{x\to 0}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2(1)=2$$

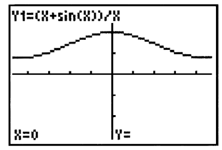
31. 

[−4.7, 4.7] by [−3.1, 3.1]

$$\lim_{x\to 0}\frac{\sin x}{2x^2-x}=-1$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{2x^2-x} &= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{2x-1}\right)\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{2x-1}\right)\\ &= (1)\left(\frac{1}{2(0)-1}\right)\\ &= -1\end{aligned}$$

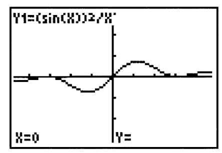
32. 

[−4.7, 4.7] by [−3.1, 3.1]

$$\lim_{x\to 0}\frac{x+\sin x}{x}=2$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0}\frac{x+\sin x}{x} &= \lim_{x\to 0}\left(1+\frac{\sin x}{x}\right)\\ &= \left(\lim_{x\to 0}1\right)+\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\\ &= 1+1\\ &= 2\end{aligned}$$

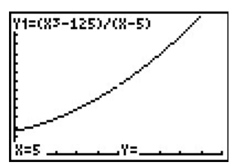
33. 

[−4.7, 4.7] by [−3.1, 3.1]

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=0$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x}{x} &= \lim_{x\to 0}\left(\sin x\cdot\frac{\sin x}{x}\right)\\ &= \left(\lim_{x\to 0}\sin x\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\\ &= (\sin 0)(1)\\ &= 0\end{aligned}$$

34. 

[0, 10] by [0, 150]

$$\lim_{x\to 5}\frac{x^3-125}{x-5}=75$$

Algebraic confirmation:

$$\begin{aligned}\lim_{x\to 5}\frac{x^3-125}{x-5} &= \lim_{x\to 5}\frac{(x-5)(x^2+5x+25)}{x-5}\\ &= \lim_{x\to 5}(x^2+5x+25)\\ &= (5)^2+5(5)+25\\ &= 75\end{aligned}$$

35. Answers will vary. One possible graph is given by the window [–4.7, 4.7] by [–15, 15] with *Xscl* = 1 and *Yscl* = 5.

36. Answers will vary. One possible graph is given by the window [–4.7, 4.7] by [–15, 15] with $Xscl = 1$ and $Yscl = 5$.

37. Since int $x = 0$ for x in (0, 1), $\lim_{x\to 0^+} \text{int } x = 0$.

38. Since int $x = -1$ for x in (–1, 0), $\lim_{x\to 0^-} \text{int } x = -1$.

39. Since int $x = 0$ for x in (0, 1), $\lim_{x\to 0.01} \text{int } x = 0$.

40. Since int $x = 1$ for x in (1, 2), $\lim_{x\to 2^-} \text{int } x = 1$.

41. Since $\frac{x}{|x|} = 1$ for $x > 0$, $\lim_{x\to 0^+} \frac{x}{|x|} = 1$.

42. Since $\frac{x}{|x|} = -1$ for $x < 0$, $\lim_{x\to 0^-} \frac{x}{|x|} = -1$.

43. **(a)** True

(b) True

(c) False, since $\lim_{x\to 0^-} f(x) = 0$.

(d) True, since both are equal to 0.

(e) True, since (d) is true.

(f) True

(g) False, since $\lim_{x\to 0} f(x) = 0$.

(h) False, $\lim_{x\to 1^-} f(x) = 1$, but $\lim_{x\to 1} f(x)$ is undefined.

(i) False, $\lim_{x\to 1^+} f(x) = 0$, but $\lim_{x\to 1} f(x)$ is undefined.

(j) False, since $\lim_{x\to 2^-} f(x) = 0$.

44. **(a)** True

(b) False, since $\lim_{x\to 2} f(x) = 1$.

(c) False, since $\lim_{x\to 2} f(x) = 1$.

(d) True

(e) True

(f) True, since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

(g) True, since both are equal to 0.

(h) True

(i) True, since $\lim_{x\to c} f(x) = 1$ for all c in (1, 3).

45. **(a)** $\lim_{x\to 3^-} f(x) = 3$

(b) $\lim_{x\to 3^+} f(x) = -2$

(c) $\lim_{x\to 3} f(x)$ does not exist, because the left- and right-hand limits are not equal.

(d) $f(3) = 1$

46. **(a)** $\lim_{t\to -4^-} g(t) = 5$

(b) $\lim_{t\to -4^+} g(t) = 2$

(c) $\lim_{t\to -4} g(t)$ does not exist, because the left- and right-hand limits are not equal.

(d) $g(-4) = 2$

47. **(a)** $\lim_{h\to 0^-} f(h) = -4$

(b) $\lim_{h\to 0^+} f(h) = -4$

(c) $\lim_{h\to 0} f(h) = -4$

(d) $f(0) = -4$

48. **(a)** $\lim_{s\to -2^-} p(s) = 3$

(b) $\lim_{s\to -2^+} p(s) = 3$

(c) $\lim_{s\to -2} p(s) = 3$

(d) $p(-2) = 3$

49. **(a)** $\lim_{x\to 0^-} F(x) = 4$

(b) $\lim_{x\to 0^+} F(x) = -3$

(c) $\lim_{x\to 0} F(x)$ does not exist, because the left- and right- hand limits are not equal.

(d) $F(0) = 4$

50. **(a)** $\lim_{x\to 2^-} G(x) = 1$

(b) $\lim_{x\to 2^+} G(x) = 1$

(c) $\lim_{x\to 2} G(x) = 1$

(d) $G(2) = 3$

51. $y_1 = \dfrac{x^2+x-2}{x-1} = \dfrac{(x-1)(x+2)}{x-1} = x+2,\ x \neq 1$

(c)

52. $y_1 = \dfrac{x^2-x-2}{x-1} = \dfrac{(x+1)(x-2)}{x-1}$

(b)

53. $y_1 = \dfrac{x^2-2x+1}{x-1} = \dfrac{(x-1)^2}{x-1} = x-1,\ x \neq 1$

(d)

54. $y_1 = \dfrac{x^2+x-2}{x+1} = \dfrac{(x-1)(x+2)}{x+1}$

(a)

55. **(a)** $\lim_{x\to 4}(g(x)+3) = \left(\lim_{x\to 4} g(x)\right) + \left(\lim_{x\to 4} 3\right)$
$= 3+3$
$= 6$

(b) $\lim_{x\to 4} x f(x) = \left(\lim_{x\to 4} x\right)\left(\lim_{x\to 4} f(x)\right)$
$= 4 \cdot 0$
$= 0$

(c) $\lim_{x\to 4} g^2(x) = \left(\lim_{x\to 4} g(x)\right)^2 = 3^2 = 9$

(d) $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1} = \dfrac{\lim_{x\to 4} g(x)}{\left(\lim_{x\to 4} f(x)\right) - \left(\lim_{x\to 4} 1\right)}$
$= \dfrac{3}{0-1}$
$= -3$

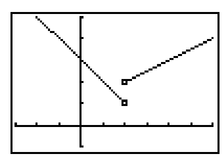
56. **(a)** $\lim_{x\to b}(f(x)+g(x))$
$= \left(\lim_{x\to b} f(x)\right) + \left(\lim_{x\to b} g(x)\right)$
$= 7 + (-3)$
$= 4$

(b) $\lim_{x\to b}(f(x)\cdot g(x))$
$= \left(\lim_{x\to b} f(x)\right)\left(\lim_{x\to b} g(x)\right)$
$= (7)\ (-3)$
$= -21$

(c) $\lim_{x\to b} 4g(x) = 4\lim_{x\to b} g(x) = 4(-3) = -12$

(d) $\lim_{x\to b} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to b} f(x)}{\lim_{x\to b} g(x)} = \dfrac{7}{-3} = -\dfrac{7}{3}$

57. **(a)**

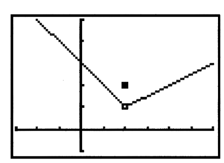


[–3, 6] by [–1, 5]

(b) $\lim_{x\to 2^+} f(x) = 2;\ \lim_{x\to 2^-} f(x) = 1$

(c) No, because the two one-sided limits are different.

58. **(a)**

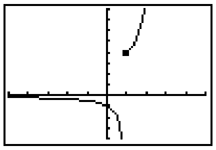


[–3, 6] by [–1, 5]

(b) $\lim_{x\to 2^+} f(x) = 1;\ \lim_{x\to 2^-} f(x) = 1$

(c) Yes; the limit is 1.

59. **(a)**

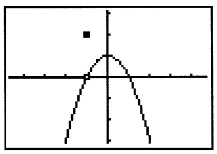


[–5, 5] by [–4, 8]

(b) $\lim_{x\to 1^+} f(x) = 4$; $\lim_{x\to 1^-} f(x)$ does not exist.

(c) No, because the left-hand limit does not exist.

60. **(a)**

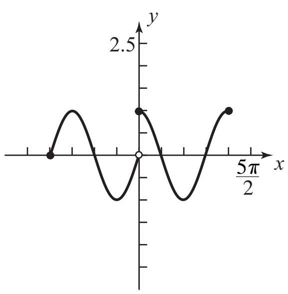


[–4.7, 4.7] by [–3.1, 3.1]

(b) $\lim_{x\to -1^+} f(x) = 0$; $\lim_{x\to -1^-} f(x) = 0$

(c) Yes; the limit is 0.

61. **(a)**

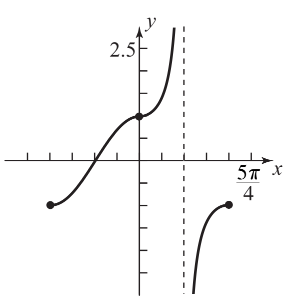


(b) $(-2\pi,\ 0)\cup(0,\ 2\pi)$

(c) $c = 2\pi$

(d) $c = -2\pi$

62. **(a)**

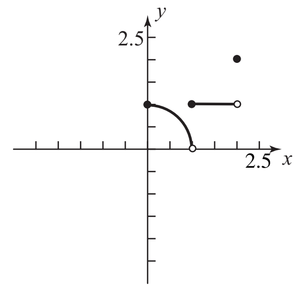


(b) $\left(-\pi, \dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2}, \pi\right)$

(c) $c = \pi$

(d) $c = -\pi$

63. **(a)**

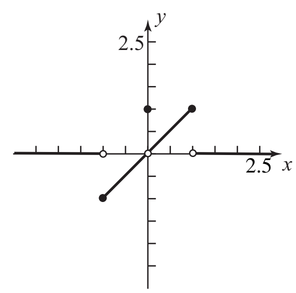


(b) $(0, 1) \cup (1, 2)$

(c) $c = 2$

(d) $c = 0$

64. **(a)**

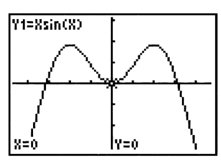


(b) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(c) None

(d) None

65.



[−4.7, 4.7] by [−3.1, 3.1]

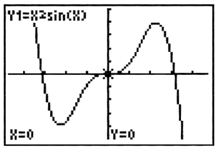
$\lim_{x\to 0} (x\sin x) = 0$

Confirm using the Sandwich Theorem, with $g(x) = -|x|$ and $h(x) = |x|$.

$|x\sin x| = |x|\cdot|\sin x| \le |x|\cdot 1 = |x|$

$-|x| \le x\sin x \le |x|$

Because $\lim_{x\to 0}\left(-|x|\right) = \lim_{x\to 0}|x| = 0,$ the Sandwich Theorem gives $\lim_{x\to 0}(x\sin x) = 0.$

66. 

$[-4.7, 4.7]$ by $[-5, 5]$

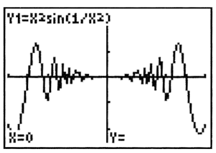
$\lim_{x\to 0} (x^2 \sin x) = 0$

Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$\left|x^2 \sin x\right| = \left|x^2\right| \cdot \left|\sin x\right| \le \left|x^2\right| \cdot 1 = x^2.$

$-x^2 \le x^2 \sin x \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0,$ the Sandwich Theorem gives $\lim_{x\to 0} (x^2 \sin x) = 0$

67. 

$[-0.5, 0.5]$ by $[-0.25, 0.25]$

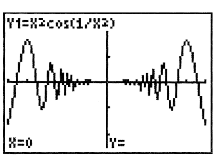
$\lim_{x\to 0}\left(x^2 \sin \frac{1}{x^2}\right) = 0$

Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$\left|x^2 \sin \frac{1}{x^2}\right| = \left|x^2\right| \cdot \left|\sin \frac{1}{x^2}\right| \le \left|x^2\right| \cdot 1 = x^2.$

$-x^2 \le x^2 \sin \frac{1}{x^2} \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0,$ the Sandwich Theorem give $\lim_{x\to 0}\left(x^2 \sin \frac{1}{x^2}\right) = 0.$

68. 

$[-0.5, 0.5]$ by $[-0.25, 0.25]$

$\lim_{x\to 0}\left(x^2 \cos \frac{1}{x^2}\right) = 0$

Confirm using the Sandwich Theorem, with $g(x) = -x^2$ and $h(x) = x^2$.

$\left|x^2 \cos \frac{1}{x^2}\right| = \left|x^2\right| \cdot \left|\cos \frac{1}{x^2}\right| \le \left|x^2\right| \cdot 1 = x^2.$

$-x^2 \le x^2 \cos \frac{1}{x^2} \le x^2$

Because $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0,$ the Sandwich Theorem give $\lim_{x\to 0}\left(x^2 \cos \frac{1}{x^2}\right) = 0.$

69. **(a)** In three seconds, the ball falls $4.9\,(3)^2 = 44.1$ m, so its average speed is $\frac{44.1}{3} = 14.7$ m/sec.

(b) The average speed over the interval from time $t = 3$ to time $3 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(3+h)^2 - 4.9(3)^2}{(3+h)-3}$$
$$= \frac{4.9(6h + h^2)}{h}$$
$$= 29.4 + 4.9h$$

Since $\lim_{h\to 0} (29.4 + 4.9h) = 29.4,$ the instantaneous speed is 29.4 m/sec.

70. **(a)** $y = gt^2$

$20 = g(4^2)$

$g = \frac{20}{16} = \frac{5}{4}$ or 1.25

(b) Average speed $= \frac{20}{4} = 5$ m/sec

(c) If the rock had not been stopped, its average speed over the interval from time $t = 4$ to time $t = 4 + h$ is

$$\frac{\Delta y}{\Delta t} = \frac{1.25(4+h)^2 - 1.25(4)^2}{(4+h)-4}$$
$$= \frac{1.25(8h + h^2)}{h}$$
$$= 10 + 1.25h$$

Since $\lim_{h\to 0} (10 + 1.25h) = 10,$ the instantaneous speed is 10 m/sec.

71. True; the definition of a limit.

72. True

$$\lim_{x\to 0}\left(\frac{x+\sin x}{x}\right)=\lim_{x\to 0}\left(1+\frac{\sin x}{x}\right)$$
$$=1+\lim_{x\to 0}\frac{\sin x}{x}$$
$$=2$$

$\sin x \approx x$ as $x \to 0$.

73. C

74. B

75. E

76. C

77. **(a)** Because the right-hand limit at zero depends only on the values of the function for positive x-values near zero.

(b) Area of $\Delta OAP = \frac{1}{2}(\text{base})(\text{height})$
$$=\frac{1}{2}(1)\,(\sin\theta)$$
$$=\frac{\sin\theta}{2}$$

Area of sector $OAP = \frac{(\text{angle})(\text{radius})^2}{2}$
$$=\frac{\theta(1)^2}{2}$$
$$=\frac{\theta}{2}$$

Area of $\Delta OAT = \frac{1}{2}(\text{base})(\text{height})$
$$=\frac{1}{2}(1)(\tan\theta)$$
$$=\frac{\tan\theta}{2}$$

(c) This is how the areas of the three regions compare.

(d) Multiply by 2 and divide by $\sin\theta$.

(e) Take reciprocals, remembering that all of the values involved are positive.

(f) The limits for $\cos\theta$ and 1 are both equal to 1. Since $\frac{\sin\theta}{\theta}$ is between them, it must also have a limit of 1.

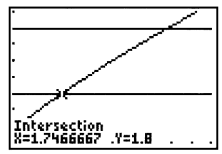
(g) $\frac{\sin(-\theta)}{-\theta}=\frac{-\sin\theta}{-\theta}=\frac{\sin\theta}{\theta}$

(h) If the function is symmetric about the y-axis, and the right-hand limit at zero is 1, then the left-hand limit at zero must also be 1.

(i) The two one-sided limits both exist and are equal to 1.

78. **(a)** The limit can be found by substitution.
$$\lim_{x\to 2} f(x)=f(2)=\sqrt{3(2)-2}=\sqrt{4}=2$$

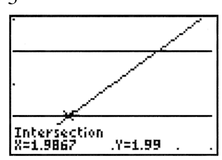
(b) The graphs of $y_1=f(x)$, $y_2=1.8$, and $y_3=2.2$ are shown.



[1.5, 2.5] by [1.5, 2.3]

The intersections of y_1 with y_2 and y_3 are at $x = 1.7467$ and $x = 2.28$, respectively, so we may choose any value of a in [1.7467, 2) (approximately) and any value of b in (2, 2.28].
One possible answer: $a = 1.75$, $b = 2.28$.

(c) The graphs of $y_1=f(x)$, $y_2=1.99$, and $y_3=2.01$ are shown.

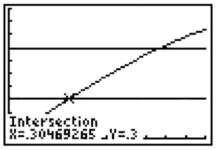


[1.97, 2.03] by [1.98, 2.02]

The intersections of y_1 with y_2 and y_3 are at $x = 1.9867$ and $x = 2.0134$, respectively, so we may choose any value of a in [1.9867, 2), and any value of b in (2, 2.0134] (approximately).
One possible answer: $a = 1.99$, $b = 2.01$.

79. **(a)** $f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

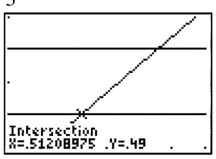
(b) The graphs of $y_1 = f(x)$, $y_2 = 0.3$, and $y_3 = 0.7$ are shown.



[0, 1] by [0, 1]

The intersections of y_1 with y_2 and y_3 are at $x = 0.3047$ and $x = 0.7754$, respectively, so we may choose any value of a in $\left[0.3047, \frac{\pi}{6}\right)$, and any value of b in $\left(\frac{\pi}{6}, 0.7754\right]$, where the interval endpoints are approximate.
One possible answer: $a = 0.305$, $b = 0.775$.

(c) The graphs of $y_1 = f(x)$, $y_2 = 0.49$, and $y_3 = 0.51$ are shown.



[0.49, 0.55] by [0.48, 0.52]

The intersections of y_1 with y_2 and y_3 are at $x = 0.5121$ and $x = 0.5352$, respectively, so we may choose any value of a in $\left[0.5121, \frac{\pi}{6}\right)$, and any value of b in $\left(\frac{\pi}{6}, 0.5352\right]$, where the interval endpoints are approximate.
One possible answer: $a = 0.513$, $b = 0.535$.

80. Line segment OP has endpoints (0, 0) and (a, a^2), so its midpoint is $\left(\frac{0+a}{2}, \frac{0+a^2}{2}\right) = \left(\frac{a}{2}, \frac{a^2}{2}\right)$ and its slope is $\frac{a^2-0}{a-0} = a$. The perpendicular bisector is the line through $\left(\frac{a}{2}, \frac{a^2}{2}\right)$ with slope $-\frac{1}{a}$, so its equation is $y = -\frac{1}{a}\left(x - \frac{a}{2}\right) + \frac{a^2}{2}$, which is equivalent to $y = -\frac{1}{a}x + \frac{1+a^2}{2}$. Thus the y-intercept is $b = \frac{1+a^2}{2}$. As the point P approaches the origin along the parabola, the value of a approaches zero. Therefore,

$$\lim_{P\to O} b = \lim_{a\to 0} \frac{1+a^2}{2} = \frac{1+0^2}{2} = \frac{1}{2}.$$

Section 2.2 Limits Involving Infinity (pp. 70–77)

Exploration 1 Exploring Theorem 5

1. Neither $\lim_{x\to\infty} f(x)$ or $\lim_{x\to\infty} g(x)$ exist. In this case, we can describe the behavior of f and g as $x \to \infty$ by writing $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We cannot apply the quotient rule because both limits must exist. However, from Example 5,
$$\lim_{x\to\infty} \frac{5x + \sin x}{x} = \lim_{x\to\infty}\left(5 + \frac{\sin x}{x}\right) = 5 + 0 = 5,$$
so the limit of the quotient exists.

2. Both f and g oscillate between 0 and 1 as $x \to \infty$, taking on each value infinitely often. We cannot apply the sum rule because neither limit exists. However,
$$\lim_{x\to\infty} (\sin^2 x + \cos^2 x) = \lim_{x\to\infty} (1) = 1,$$
so the limit of the sum exists.

3. The limit of f and g as $x \to \infty$ do not exist, so we cannot apply the difference rule to $f - g$. We can say that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We can write the difference as $f(x) - g(x) = \ln(2x) - \ln(x+1) = \ln\frac{2x}{x+1}$. We can use graphs or tables to convince ourselves that this limit is equal to ln 2.

4. The fact that the limits of f and g as $x \to \infty$ do not exist does not necessarily mean that the limits of $f + g$, $f - g$ or $\frac{f}{g}$ do not exist, just that Theorem 5 cannot be applied.

Quick Review 2.2

1. $y = 2x - 3$

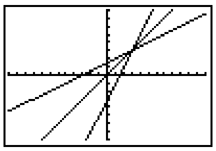
$y + 3 = 2x$

$\dfrac{y+3}{2} = x$

Interchange x and y.

$\dfrac{x+3}{2} = y$

$f^{-1}(x) = \dfrac{x+3}{2}$



[–12, 12] by [–8, 8]

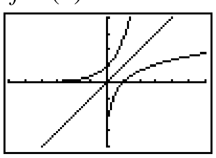
2. $y = e^x$

$\ln y = x$

Interchange x and y.

$\ln x = y$

$f^{-1}(x) = \ln x$



[–6, 6] by [–4, 4]

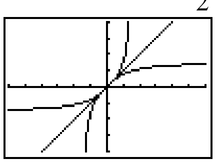
3. $y = \tan^{-1} x$

$\tan y = x,\ -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Interchange x and y.

$\tan x = y, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$f^{-1}(x) = \tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$



[–6, 6] by [–4, 4]

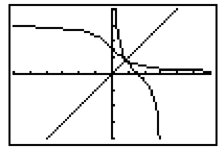
4. $y = \cot^{-1} x$

$\cot y = x,\ 0 < y < \pi$

Interchange x and y.

$\cot x = y,\ 0 < x < \pi$

$f^{-1}(x) = \cot x,\ 0 < x < \pi$



[–6, 6] by [–4, 4]

5.

$$\begin{array}{r|l} & \frac{2}{3} \\ \hline 3x^3 + 4x - 5 & 2x^3 - 3x^2 + x - 1 \\ & \underline{2x^3 + 0x^2 + \frac{8}{3}x - \frac{10}{3}} \\ & \quad -3x^2 - \frac{5}{3}x + \frac{7}{3} \end{array}$$

$q(x) = \dfrac{2}{3}$

$r(x) = -3x^2 - \dfrac{5}{3}x + \dfrac{7}{3}$

6.

$$\begin{array}{r|l} & 2x^2 + 2x + 1 \\ \hline x^3 - x^2 + 1 & 2x^5 + 0x^4 - x^3 + 0x^2 + x - 1 \\ & \underline{2x^5 - 2x^4 + 0x^3 + 2x^2} \\ & \quad 2x^4 - x^3 - 2x^2 + x - 1 \\ & \quad \underline{2x^4 - 2x^3 + 0x^2 + 2x} \\ & \qquad x^3 - 2x^2 - x - 1 \\ & \qquad \underline{x^3 - x^2 + 0x + 1} \\ & \qquad\quad -x^2 - x - 2 \end{array}$$

$q(x) = 2x^2 + 2x + 1$

$r(x) = -x^2 - x - 2$

7. **(a)** $f(-x) = \cos(-x) = \cos x$

(b) $f\left(\dfrac{1}{x}\right) = \cos\left(\dfrac{1}{x}\right)$

8. **(a)** $f(-x) = e^{-(-x)} = e^x$

(b) $f\left(\dfrac{1}{x}\right) = e^{-1/x}$

9. **(a)** $f(-x) = \dfrac{\ln(-x)}{-x} = -\dfrac{\ln(-x)}{x}$

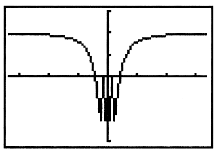
(b) $f\left(\dfrac{1}{x}\right) = \dfrac{\ln\frac{1}{x}}{\frac{1}{x}} = x \ln x^{-1} = -x \ln x$

10. **(a)** $$f(-x) = \left(-x + \frac{1}{-x}\right)\sin(-x)$$
$$= -\left(x + \frac{1}{x}\right)(-\sin x)$$
$$= \left(x + \frac{1}{x}\right)\sin x$$

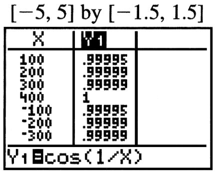
(b) $$f\left(\frac{1}{x}\right) = \left(\frac{1}{x} + \frac{1}{\frac{1}{x}}\right)\sin\left(\frac{1}{x}\right)$$
$$= \left(\frac{1}{x} + x\right)\sin\left(\frac{1}{x}\right)$$

Section 2.2 Exercises

1.



[−5, 5] by [−1.5, 1.5]



X	Y1
100	.99995
200	.99999
300	.99999
400	1
-100	.99995
-200	.99999
-300	.99999

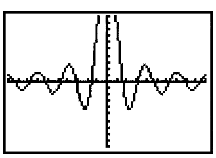
Y1=cos(1/X)

(a) $\lim_{x\to\infty} f(x) = 1$

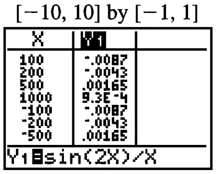
(b) $\lim_{x\to-\infty} f(x) = 1$

(c) $y = 1$

2.



[−10, 10] by [−1, 1]



X	Y1
100	-.0087
200	-.0043
500	.00165
1000	9.3E-4
-100	-.0087
-200	-.0043
-500	.00165

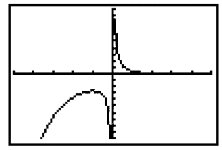
Y1=sin(2X)/X

(a) $\lim_{x\to\infty} f(x) = 0$

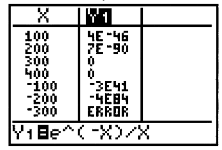
(b) $\lim_{x\to-\infty} f(x) = 0$

(c) $y = 0$

3.



[−5, 5] by [−10, 10]



X	Y1
100	4E-46
200	7E-90
300	0
400	0
-100	-3E41
-200	-4E84
-300	ERROR

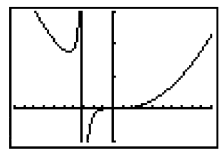
Y1=e^(-X)/X

(a) $\lim_{x\to\infty} f(x) = 0$

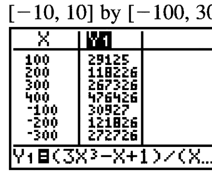
(b) $\lim_{x\to-\infty} f(x) = -\infty$

(c) $y = 0$

4.



[−10, 10] by [−100, 300]



X	Y1
100	29125
200	118226
300	267326
400	476426
-100	30927
-200	121826
-300	272726

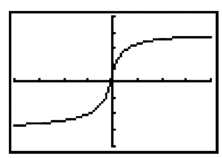
Y1=(3X³-X+1)/(X...

(a) $\lim_{x\to\infty} f(x) = \infty$

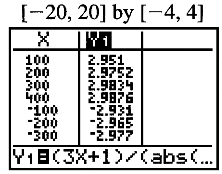
(b) $\lim_{x\to-\infty} f(x) = \infty$

(c) No horizontal asymptotes.

5.



[−20, 20] by [−4, 4]



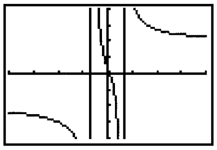
X	Y1
100	2.951
200	2.9752
300	2.9834
400	2.9876
-100	-2.931
-200	-2.965
-300	-2.977

Y1=(3X+1)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 3$

(b) $\lim_{x\to-\infty} f(x) = -3$

(c) $y = 3, y = -3$

6. 

[−20, 20] by [−4, 4]

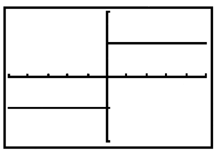
X	Y1	
100	2.0515	
200	2.0254	
500	2.0101	
1000	2.005	
-100	-2.072	
-200	-2.036	
-500	-2.014	

Y1=(2X−1)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 2$

(b) $\lim_{x\to-\infty} f(x) = -2$

(c) $y = 2, y = -2$

7. 

[−5, 5] by [−2, 2]

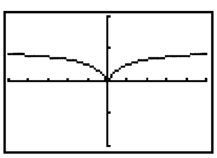
X	Y1	
100	1	
200	1	
500	1	
1000	1	
-100	-1	
-200	-1	
-500	-1	

Y1=X/abs(X)

(a) $\lim_{x\to\infty} f(x) = 1$

(b) $\lim_{x\to-\infty} f(x) = -1$

(c) $y = 1, y = -1$

8. 

[−5, 5] by [−2, 2]

X	Y1	
100	.9901	
200	.99502	
500	.998	
1000	.999	
-100	.9901	
-200	.99502	
-500	.998	

Y1=abs(X)/(abs(...

(a) $\lim_{x\to\infty} f(x) = 1$

(b) $\lim_{x\to-\infty} f(x) = 1$

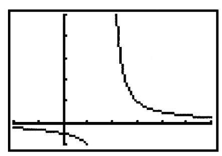
(c) $y = 1$

9. $0 \le 1 - \cos x \le 2$. So, for $x > 0$ we have
$0 \le \frac{1-\cos x}{x^2} \le \frac{2}{x^2}$.
By the Sandwich Theorem,
$$0 = \lim_{x\to\infty} (0) = \lim_{x\to\infty} \frac{1-\cos x}{x^2} = \lim_{x\to\infty} \frac{2}{x^2} = 0.$$

10. $0 \le 1 - \cos x \le 2$. So, for $x > 0$ we have
$0 \le \frac{1-\cos x}{x^2} \le \frac{2}{x^2}$. By the Sandwich Theorem,
$$0 = \lim_{x\to-\infty} (0) = \lim_{x\to-\infty} \frac{1-\cos x}{x^2} = \lim_{x\to-\infty} \frac{2}{x^2} = 0.$$

11. $-1 \le \sin x \le 1$, so for $x < 0$, $\frac{1}{x} \le \frac{\sin x}{x} \le -\frac{1}{x}$.
Therefore
$$\lim_{x\to-\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{\sin x}{x} = \lim_{x\to-\infty} \left(-\frac{1}{x}\right) = 0$$
by the Sandwich Theorem.

12. $-1 \le \sin(x^2) \le 1$, so for $x > 0$,
$$-\frac{1}{x} \le \frac{\sin(x^2)}{x} \le \frac{1}{x}$$
$$\lim_{x\to\infty} \frac{-1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} \frac{\sin(x^2)}{x}.$$

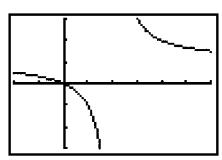
13. 

[−2, 6] by [−1, 5]

X	Y1	
2.8	1.25	
2.4	2.5	
2.2	5	
2.1	10	
2.01	100	
2.001	1000	
2.0001	10000	

Y1=1/(X−2)

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$

14. 

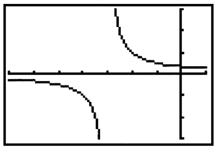
[−2, 6] by [−3, 3]

X	Y1
1.2	-1.5
1.6	-4
1.8	-9
1.9	-19
1.99	-199
1.999	-1999
1.9999	-19999

Y1=X/(X-2)

$$\lim_{x\to 2^-} \frac{x}{x-2} = -\infty$$

15.



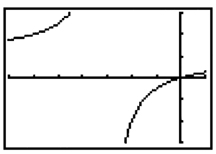
$[-7, 1]$ by $[-3, 3]$

X	Y1
-3.8	-1.25
-3.4	-2.5
-3.3	-3.333
-3.2	-5
-3.1	-10
-3.01	-100
-3.001	-1000

Y1=1/(X+3)

$$\lim_{x\to -3^-} \frac{1}{x+3} = -\infty$$

16.



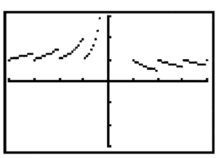
$[-7, 1]$ by $[-3, 3]$

X	Y1
-2.2	-2.75
-2.6	-6.5
-2.7	-9
-2.8	-14
-2.9	-29
-2.99	-299
-2.999	-2999

Y1=X/(X+3)

$$\lim_{x\to -3^+} \frac{x}{x+3} = -\infty$$

17.



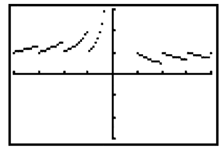
$[-4, 4]$ by $[-3, 3]$

X	Y1
.8	0
.4	0
.2	0
.1	0
.01	0
.001	0
1E-4	0

Y1=int(X)/X

$$\lim_{x\to 0^+} \frac{\operatorname{int} x}{x} = 0$$

18.



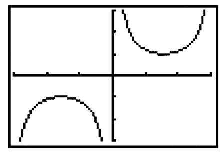
$[-4, 4]$ by $[-3, 3]$

X	Y1
-.8	1.25
-.4	2.5
-.2	5
-.1	10
-.01	100
-.001	1000
-1E-4	10000

Y1=int(X)/X

$$\lim_{x\to 0^-} \frac{\operatorname{int} x}{x} = \infty$$

19.



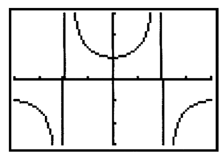
$[-3, 3]$ by $[-3, 3]$

X	Y1
.8	1.394
.4	2.5679
.2	5.0335
.1	10.017
.01	100
.001	1000
1E-4	10000

Y1=1/sin(X)

$$\lim_{x\to 0^+} \csc x = \infty$$

20



$[-\pi, \pi]$ by $[-3, 3]$

X	Y1
1.6	-34.25
1.59	-52.08
1.58	-108.7
1.576	-192.2
1.572	-830.8
1.571	-4910
1.5708	-2.7E5

Y1=1/cos(X)

$$\lim_{x\to \left(\frac{\pi}{2}\right)^+} \sec x = -\infty$$

21.

$$\begin{aligned} y &= \left(2-\frac{x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right) \\ &= \left(\frac{2(x+1)-x}{x+1}\right)\left(\frac{x^2}{5+x^2}\right) \\ &= \left(\frac{x+2}{x+1}\right)\left(\frac{x^2}{5+x^2}\right) \\ &= \frac{x^3+2x^2}{x^3+x^2+5x+5} \end{aligned}$$

An end behavior model for y is $\dfrac{x^3}{x^3} = 1.$

$\lim_{x\to\infty} y = \lim_{x\to\infty} 1 = 1$
$\lim_{x\to-\infty} y = \lim_{x\to-\infty} 1 = 1$

22. $y = \left(\dfrac{2}{x} + 1\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$
$= \left(\dfrac{2 + x}{x}\right)\left(\dfrac{5x^2 - 1}{x^2}\right)$
$= \dfrac{5x^3 + 10x^2 - x - 2}{x^3}$

An end behavior model for y is $\dfrac{5x^3}{x^3} = 5.$

$\lim_{x\to\infty} y = \lim_{x\to\infty} 5 = 5$
$\lim_{x\to-\infty} y = \lim_{x\to-\infty} 5 = 5$

23. Use the method of Example 9 in the text.

$\lim_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x\to 0^+} \dfrac{\cos x}{1 + x} = \dfrac{\cos(0)}{1 + 0} = \dfrac{1}{1} = 1$

$\lim_{x\to-\infty} \dfrac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \lim_{x\to 0^-} \dfrac{\cos x}{1 + x} = \dfrac{\cos(0)}{1 + 0} = \dfrac{1}{1} = 1$

24. Note that $y = \dfrac{2x + \sin x}{x} = 2 + \dfrac{\sin x}{x}.$

So, $\lim_{x\to\infty} y = \lim_{x\to\infty} 2 + \lim_{x\to\infty} \dfrac{\sin x}{x} = 2 + 0 = 2.$
Similarly, $\lim_{x\to-\infty} y = 2.$

25. Use $y = \dfrac{\sin x}{2x^2 + x} = \dfrac{\sin x}{x} \cdot \dfrac{1}{2x + 1}$

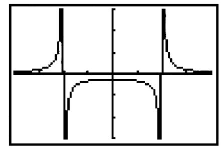
$\lim_{x\to\pm\infty} \dfrac{\sin x}{x} = 0$

$\lim_{x\to\pm\infty} \dfrac{1}{2x + 1} = 0$

So, $\lim_{x\to\infty} y = 0$ and $\lim_{x\to-\infty} y = 0.$

26. $y = \dfrac{1}{2}\dfrac{\sin x}{x} + \dfrac{1}{x}\dfrac{\sin x}{x}$
So, $\lim_{x\to\infty} y = 0$ and $\lim_{x\to-\infty} y = 0.$

27.

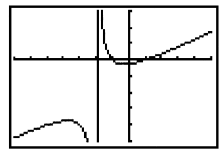


[−4, 4] by [−3, 3]

(a) $x = -2, x = 2$

(b) Left-hand limit at –2 is ∞.
Right-hand limit at –2 is –∞.
Left-hand limit at 2 is –∞.
Right-hand limit at 2 is ∞.

28.

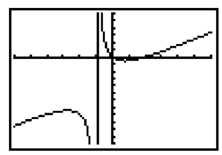


[−7, 5] by [−5, 3]

(a) $x = -2$

(b) Left-hand limit at –2 is –∞.
Right-hand limit at –2 is ∞.

29.

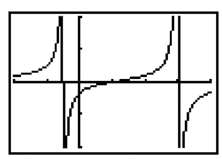


[−6, 6] by [−12, 6]

(a) $x = -1$

(b) Left-hand limit at –1 is –∞.
Right-hand limit at –1 is ∞.

30.

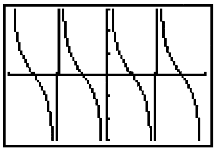


[−2, 4] by [−2, 2]

(a) $x = -\dfrac{1}{2},\ x = 3$

(b) Left-hand limit at $-\dfrac{1}{2}$ is ∞.

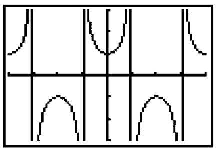
Right-hand limit at $-\dfrac{1}{2}$ is –∞.
Left-hand limit at 3 is ∞.
Right-hand limit at 3 is –∞.

31. 

[−2π, 2π] by [−3, 3]

(a) $x = k\pi,\ k$ any integer

(b) at each vertical asymptote:
Left-hand limit is $-\infty$.
Right-hand limit is ∞.

32. 

[−2π, 2π] by [−3, 3]

(a) $x = \frac{\pi}{2} + n\pi,\ n$ any integer

(b) If n is even:
Left-hand limit is ∞.
Right-hand limit is $-\infty$.
If n is odd:
Left-hand limit is $-\infty$.
Right-hand limit is ∞.

33. $f(x) = \frac{\tan x}{\sin x} = \frac{1}{\sin x}\frac{\sin x}{\cos x} = \frac{1}{\cos x}$

$\cos x = 0$ at: $a = (4k+1)\frac{\pi}{2}$ and $b = (4k+3)\frac{\pi}{2}$
where k is any real integer.
$\lim_{x\to a^-} f(x) = \infty,\ \lim_{x\to a^+} f(x) = -\infty,$
$\lim_{x\to b^-} f(x) = -\infty,\ \lim_{x\to b^+} f(x) = \infty.$

34. $f(x) = \frac{\cot x}{\cos x} = \frac{\cos x}{\sin x}\frac{1}{\cos x} = \frac{1}{\sin x}$
$\sin x = 0$ at $a = 2k\pi$ and $b = (2k+1)\pi$ where k is any real integer.
$\lim_{x\to a^-} f(x) = -\infty,\ \lim_{x\to a^+} f(x) = \infty,$
$\lim_{x\to b^-} f(x) = \infty,\ \lim_{x\to b^+} f(x) = -\infty.$

35. An end behavior model is $\frac{2x^3}{x} = 2x^2$. (a)

36. An end behavior model is $\frac{x^5}{2x^2} = 0.5x^3$. (c)

37. An end behavior model is $\frac{2x^4}{-x} = -2x^3$. (d)

38. An end behavior model is $\frac{x^4}{-x^2} = -x^2$. (b)

39. **(a)** $3x^2$

(b) None

40. **(a)** $-4x^3$

(b) None

41. **(a)** $\frac{x}{2x^2} = \frac{1}{2x}$

(b) $y = 0$

42. **(a)** $\frac{3x^2}{x^2} = 3$

(b) $y = 3$

43. **(a)** $\frac{4x^3}{x} = 4x^2$

(b) None

44. **(a)** $\frac{-x^4}{x^2} = -x^2$

(b) None

45. **(a)** The function $y = e^x$ is a right end behavior model because

$$\lim_{x\to\infty} \frac{e^x - 2x}{e^x} = \lim_{x\to\infty}\left(1 - \frac{2x}{e^x}\right) = 1 - 0 = 1.$$

(b) The function $y = -2x$ is a left end behavior model because

$$\lim_{x\to-\infty} \frac{e^x - 2x}{-2x} = \lim_{x\to-\infty}\left(-\frac{e^x}{2x} + 1\right)$$
$$= 0 + 1$$
$$= 1.$$

46. **(a)** The function $y = x^2$ is a right end behavior model because

$$\lim_{x\to\infty} \frac{x^2+e^{-x}}{x^2} = \lim_{x\to\infty}\left(1+\frac{e^{-x}}{x^2}\right) = 1+0 = 1.$$

(b) The function $y = e^{-x}$ is a left end behavior model because

$$\lim_{x\to-\infty} \frac{x^2+e^{-x}}{e^{-x}} = \lim_{x\to-\infty}\left(\frac{x^2}{e^{-x}}+1\right)$$
$$= \lim_{x\to-\infty}(x^2e^x+1)$$
$$= 0+1$$
$$= 1.$$

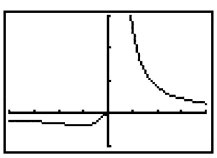
47. **(a, b)** The function $y = x$ is both a right end behavior model and a left end behavior model because

$$\lim_{x\to\pm\infty}\left(\frac{x+\ln|x|}{x}\right) = \lim_{x\to\pm\infty}\left(1+\frac{\ln|x|}{x}\right)$$
$$= 1+0$$
$$= 1.$$

48. **(a, b)** The function $y = x^2$ is both a right end behavior model and a left end behavior model because

$$\lim_{x\to\pm\infty}\left(\frac{x^2+\sin x}{x^2}\right) = \lim_{x\to\pm\infty}\left(1+\frac{\sin x}{x^2}\right) = 1.$$

49.



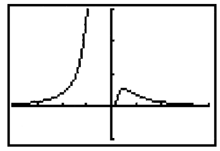
[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x}e^{1/x}$ is shown.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$$

50.



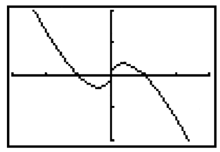
[−4, 4] by [−1, 3]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{1}{x^2}e^{-1/x}$ is shown.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = \infty$$

51.



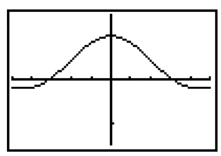
[−3, 3] by [−2, 2]

The graph of $y = f\left(\frac{1}{x}\right) = x\ln\left|\frac{1}{x}\right|$ is shown.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 0$$

52.



[−5, 5] by [−1.5, 1.5]

The graph of $y = f\left(\frac{1}{x}\right) = \frac{\sin x}{x}$ is shown.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right) = 1$$

53. **(a)** $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(-1) = -1$

(c) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{1}{x} = -\infty$

(d) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(-1) = -1$

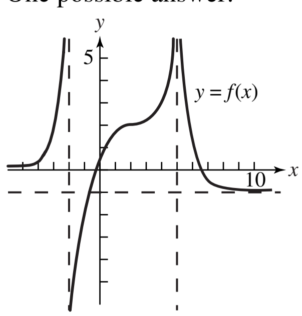
54. **(a)** $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-2}{x-1} = \lim_{x\to-\infty} \frac{x}{x} = 1$

(b) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} = 0$

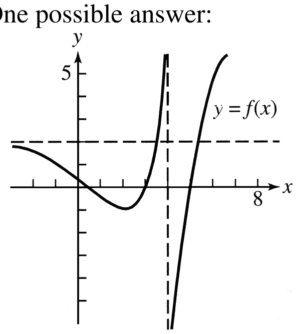
(c) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-2}{x-1} = \frac{0-2}{0-1} = 2$

(d) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2} = \infty$

55. One possible answer:



56. One possible answer:



57. Note that $\dfrac{\frac{f_1(x)}{f_2(x)}}{\frac{g_1(x)}{g_2(x)}} = \dfrac{f_1(x)g_2(x)}{g_1(x)f_2(x)} = \dfrac{\frac{f_1(x)}{g_2(x)}}{\frac{f_2(x)}{g_2(x)}}$.

As x becomes large, $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ both approach 1. Therefore, using the above equation, $\dfrac{\frac{f_1}{f_2}}{\frac{g_1}{g_2}}$ must also approach 1.

58. Yes. The limit of $(f+g)$ will be the same as the limit of g. This is because adding numbers that are very close to a given real number L will not have a significant effect on the value of $(f+g)$ since the values of g are becoming arbitrarily large.

59. True. For example, $f(x) = \frac{x}{\sqrt{x^2+1}}$ has $y = \pm 1$ as horizontal asymptotes.

60. False; consider $f(x) = \frac{1}{x}$.

61. A; $\lim_{x\to 2^-}(x-2)$ approaches zero from the left, so $\lim_{x\to 2^-} \frac{x}{x-2}$ approaches $-\infty$.

62. E; $\lim_{x\to 0} \frac{\cos(2x)}{x}$ is undefined because $\frac{\cos(2x)}{x}$ has a vertical asymptote at $x = 0$.

63. C; let $t = 3x$.
Then $t \to 0$ as $x \to 0$; and

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)}{x} &= \lim_{t\to 0} \frac{\sin t}{t/3}\\ &= 3 \cdot \lim_{t\to 0} \frac{\sin t}{t}\\ &= 3 \cdot 1\\ &= 3\end{aligned}$$

64. D; $\frac{2x^3}{x^3} = 2$

65. **(a)** Note that $fg = f(x)g(x) = 1$.
$f \to -\infty$ as $x \to 0^-$, $f \to \infty$ as $x \to 0^+$,
$g \to 0, fg \to 1$

(b) Note that $fg = f(x)g(x) = -8$.
$f \to \infty$ as $x \to 0^-$, $f \to -\infty$ as $x \to 0^+$,
$g \to 0, fg \to -8$

(c) Note that $fg = f(x)g(x) = 3(x-2)^2$.
$f \to -\infty$ as $x \to 2^-$, $f \to \infty$ as $x \to 2^+$,
$g \to 0, fg \to 0$

(d) Note that $fg = f(x)g(x) = \frac{5}{(x-3)^2}$.
$f \to \infty, g \to 0, fg \to \infty$

(e) Nothing—you need more information to decide.

66. **(a)** This follows from $x - 1 < \text{int } x \le x$, which is true for all x. Dividing by x gives the result.

(b, c) Since $\lim\limits_{x\to\pm\infty} \dfrac{x-1}{x} = \lim\limits_{x\to\pm\infty} 1 = 1$, the Sandwich Theorem gives

$$\lim_{x\to\infty} \frac{\text{int } x}{x} = \lim_{x\to-\infty} \frac{\text{int } x}{x} = 1.$$

68. This is because as x approaches infinity, $\sin x$ continues to oscillate between 1 and –1 and doesn't approach any given real number.

69. $\lim\limits_{x\to\infty} \dfrac{\ln x^2}{\ln x} = 2$, because $\dfrac{\ln x^2}{\ln x} = \dfrac{2\ln x}{\ln x} = 2$.

70. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\log x} = \ln(10)$, since

$$\frac{\ln x}{\log x} = \frac{\ln x}{\frac{(\ln x)}{(\ln 10)}} = \ln 10.$$

71. $\lim\limits_{x\to\infty} \dfrac{\ln(x+1)}{\ln x} = 1$

Since

$$\ln(x+1) = \ln\left[x\left(1+\frac{1}{x}\right)\right] = \ln x + \ln\left(1+\frac{1}{x}\right),$$

$$\frac{\ln(x+1)}{\ln x} = \frac{\ln x + \ln\left(1+\frac{1}{x}\right)}{\ln x} = 1 + \frac{\ln\left(1+\frac{1}{x}\right)}{\ln x}$$

But as $x \to \infty$, $1+\dfrac{1}{x}$ approaches 1, so $\ln\left(1+\dfrac{1}{x}\right)$ approaches $\ln(1) = 0$. Also, as $x \to \infty$, $\ln x$ approaches infinity. This means the second term above approaches 0 and the limit is 1.

Quick Quiz Sections 2.1 and 2.2

1. D; $$\lim_{x\to3}\left(\frac{x^2-x-6}{x-3}\right) = \lim_{x\to3}\frac{(x-3)(x+2)}{x-3} = \lim_{x\to3}(x+2) = 3+2 = 5$$

2. A; $\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} \dfrac{5}{x+1} = \dfrac{5}{2+1} = \dfrac{5}{3}$

3. E; $\dfrac{3x^3}{2x^3} = \dfrac{3}{2}$

4. **(a)** Domain: $(-\infty, 0)\cup(0, \infty)$
Range: $(-\infty, \infty)$

(b) $f(-x) = \dfrac{\cos(-x)}{-x} = \dfrac{\cos x}{-x} = -f(x)$, so f is odd.

(c) $\lim\limits_{x\to\infty} \dfrac{\cos x}{x}, -1 \le \cos x \le 1$

$\lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$

(d) For all $x > 0, -1 \le \cos x \le 1$.

Therefore, $-\dfrac{1}{x} \le \dfrac{\cos x}{x} \le \dfrac{1}{x}$.

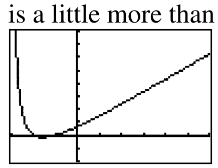
Since $\lim\limits_{x\to\infty}\left(-\dfrac{1}{x}\right) = \lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right) = 0$, it follows by the Sandwich Theorem that $\lim\limits_{x\to\infty} \dfrac{\cos x}{x} = 0$.

Section 2.3 Continuity (pp. 78–86)

Exploration 1 Removing a Discontinuity

1. $x^2 - 9 = (x-3)(x+3)$. The domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ or all $x \ne \pm3$.

2. It appears that the limit of f as $x \to 3$ exists and is a little more than 3.



[−3, 6] by [−2, 8]

3. $f(3)$ should be defined as $\dfrac{10}{3}$.

4. $x^3 - 7x - 6 = (x-3)(x+1)(x+2)$,

$x^2 - 9 = (x-3)(x+3)$, so

$f(x) = \dfrac{(x+1)(x+2)}{x+3}$ for $x \ne 3$.

Thus, $\lim\limits_{x\to3} \dfrac{(x+1)(x+2)}{x+3} = \dfrac{20}{6} = \dfrac{10}{3}$.

5. $\lim_{x \to 3} g(x) = \frac{10}{3} = g(3)$, so g is continuous at $x = 3$.

Quick Review 2.3

1. $\lim_{x \to -1} \frac{3x^2 - 2x + 1}{x^3 + 4} = \frac{3(-1)^2 - 2(-1) + 1}{(-1)^3 + 4} = \frac{6}{3} = 2$

2. **(a)** $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \text{int}(x) = -2$

(b) $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} f(x) = -1$

(c) $\lim_{x \to -1} f(x)$ does not exist, because the left- and right-hand limits are not equal.

(d) $f(-1) = \text{int}(-1) = -1$

3. **(a)**
$$\begin{aligned}\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x^2 - 4x + 5)\\ &= 2^2 - 4(2) + 5\\ &= 1\end{aligned}$$

(b) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4 - x) = 4 - 2 = 2$

(c) $\lim_{x \to 2} f(x)$ does not exist, because the left- and right-hand limits are not equal.

(d) $f(2) = 4 - 2 = 2$

4.
$$\begin{aligned}(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x} + 1\right) &= \frac{2\left(\frac{1}{x} + 1\right) - 1}{\left(\frac{1}{x} + 1\right) + 5}\\ &= \frac{2(1 + x) - x}{(1 + x) + 5x}\\ &= \frac{x + 2}{6x + 1},\ x \neq 0\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g\left(\frac{2x - 1}{x + 5}\right) &= \frac{1}{\frac{2x-1}{x+5}} + 1\\ &= \frac{x + 5}{2x - 1} + \frac{2x - 1}{2x - 1}\\ &= \frac{3x + 4}{2x - 1},\ x \neq -5\end{aligned}$$

5. Note that
$\sin x^2 = (g \circ f)(x) = g(f(x)) = g(x^2)$.
Therefore: $g(x) = \sin x,\ x \geq 0$
$(f \circ g)(x) = f(g(x)) = f(\sin x) = (\sin x^2)$ or $\sin^2 x,\ x \geq 0$

6. Note that
$\frac{1}{x} = (g \circ f)(x) = g(f(x)) = \sqrt{f(x) - 1}$.

Therefore, $\sqrt{f(x) - 1} = \frac{1}{x}$ for $x > 0$. Squaring both sides gives $f(x) - 1 = \frac{1}{x^2}$. Therefore,

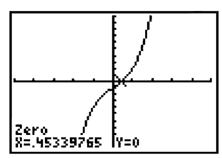
$f(x) = \frac{1}{x^2} + 1,\ x > 0$.

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= \frac{1}{(\sqrt{x - 1})^2} + 1\\ &= \frac{1}{x - 1} + 1\\ &= \frac{1 + x - 1}{x - 1}\\ &= \frac{x}{x - 1},\ x > 1\end{aligned}$$

7.
$$\begin{aligned}2x^2 + 9x - 5 &= 0\\ (2x - 1)(x + 5) &= 0\end{aligned}$$

Solutions: $x = \frac{1}{2},\ x = -5$

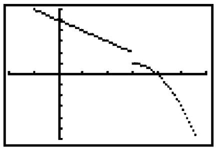
8.



$[-5, 5]$ by $[-10, 10]$

Solution: $x = 0.453$

9. For $x \leq 3$, $f(x) = 4$ when $5 - x = 4$, which gives $x = 1$. (Note that this value is, in fact, ≤ 3.)
For $x > 3$, $f(x) = 4$ when $-x^2 + 6x - 8 = 4$, which gives $x^2 - 6x + 12 = 0$. The discriminant of this equation is $b^2 - 4ac = (-6)^2 - 4(1)(12) = -12$. Since the discriminant is negative, the quadratic equation has no solution. The only solution to the original equation is $x = 1$.

10. 

[−2.7, 6.7] by [−6, 6]

A graph of $f(x)$ is shown. The range of $f(x)$ is $(-\infty, 1) \cup [2, \infty)$. The values of c for which $f(x) = c$ has no solution are the values that are excluded from the range. Therefore, c can be any value in $[1, 2)$.

Section 2.3 Exercises

1. The function $y = \dfrac{1}{(x+2)^2}$ is continuous because it is a quotient of polynomials, which are continuous. Its only point of discontinuity occurs where it is undefined. There is an infinite discontinuity at $x = -2$.

2. The function $y = \dfrac{x+1}{x^2-4x+3}$ is continuous because it is a quotient of polynomials, which are continuous. Its only points of discontinuity occur where it is undefined, that is, where the denominator $x^2 - 4x + 3 = (x - 1)(x - 3)$ is zero. There are infinite discontinuities at $x = 1$ and at $x = 3$.

3. The function $y = \dfrac{1}{x^2+1}$ is continuous because it is a quotient of polynomials, which are continuous. Furthermore, the domain is all real numbers because the denominator, $x^2 + 1$, is never zero. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

4. The function $y = |x-1|$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = |x|$ and $g(x) = x - 1$, so it is continuous. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

5. The function $y = \sqrt{2x+3}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \sqrt{x}$ and $g(x) = 2x + 3$, so it is continuous. Its points of discontinuity are the points not in the domain, i.e., all $x < -\dfrac{3}{2}$.

6. The function $y = \sqrt[3]{2x-1}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \sqrt[3]{x}$ and $g(x) = 2x - 1$, so it is continuous. Since the function is continuous and has domain $(-\infty, \infty)$, there are no points of discontinuity.

7. The function $y = \dfrac{|x|}{x}$ is equivalent to

$$y = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}.$$

It has a jump discontinuity at $x = 0$.

8. The function $y = \cot x$ is equivalent to $y = \dfrac{\cos x}{\sin x}$, a quotient of continuous functions, so it is continuous. Its only points of discontinuity occur where it is undefined. It has infinite discontinuities at $x = k\pi$ for all integers k.

9. The function $y = e^{1/x}$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = e^x$ and $g(x) = \dfrac{1}{x}$, so it is continuous. Its only point of discontinuity occurs at $x = 0$, where it is undefined. Since $\lim_{x \to 0^+} e^{1/x} = \infty$, this may be considered an infinite discontinuity.

10. The function $y = \ln(x + 1)$ is a composition $(f \circ g)(x)$ of the continuous functions $f(x) = \ln x$ and $g(x) = x + 1$, so it is continuous. Its points of discontinuity are the points not in the domain, i.e., $x \le -1$.

11. **(a)** Yes, $f(-1) = 0$.

(b) Yes, $\lim_{x \to -1^+} f(x) = 0$.

(c) Yes

(d) Yes, since −1 is a left endpoint of the domain of f and $\lim_{x \to -1^+} f(x) = f(-1)$, f is continuous at $x = -1$.

12. **(a)** Yes, $f(1) = 1$.

(b) Yes, $\lim_{x \to 1} f(x) = 2$.

(c) No

(d) No

13. **(a)** No

(b) No, since $x = 2$ is not in the domain.

14. Everywhere in $[-1, 3)$ except for $x = 0, 1, 2$.

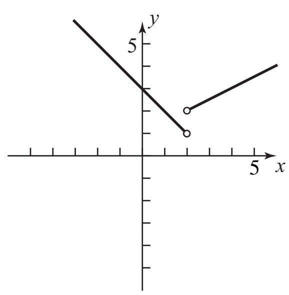
15. Since $\lim_{x\to 2} f(x) = 0$, we should assign $f(2) = 0$.

16. Since $\lim_{x\to 1} f(x) = 2$, we should reassign $f(1) = 2$.

17. No, because the right-hand and left-hand limits are not the same at zero.

18. Yes, Assign the value 0 to $f(3)$. Since 3 is a right endpoint of the extended function and $\lim_{x\to 3^-} f(x) = 0$, the extended function is continuous at $x = 3$.

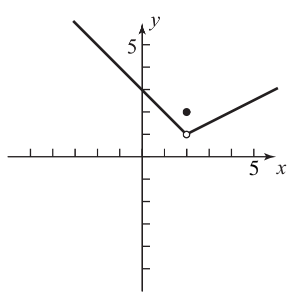
19.



(a) $x = 2$

(b) Not removable, the one-sided limits are different.

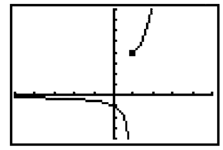
20.



(a) $x = 2$

(b) Removable, assign the value 1 to $f(2)$.

21.

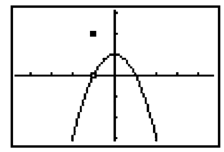


$[-5, 5]$ by $[-4, 8]$

(a) $x = 1$

(b) Not removable, it's an infinite discontinuity.

22.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$

(a) $x = -1$

(b) Removable, assign the value 0 to $f(-1)$.

23. **(a)** All points not in the domain along with $x = 0, 1$

(b) $x = 0$ is a removable discontinuity, assign $f(0) = 0$.
$x = 1$ is not removable, the one-sided limits are different.

24. **(a)** All points not in the domain along with $x = 1, 2$

(b) $x = 1$ is not removable, the one-sided limits are different.
$x = 2$ is a removable discontinuity, assign $f(2) = 1$.

25. For $x \neq -3$,

$$f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x-3.$$

The extended function is $y = x - 3$.

26. For $x \neq 1$,

$$f(x) = \frac{x^3-1}{x^2-1} = \frac{(x-1)(x^2+x+1)}{(x+1)(x-1)} = \frac{x^2+x+1}{x+1}.$$

The extended function is $y = \dfrac{x^2+x+1}{x+1}$.

27. Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the extended function is

$$y = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

28. Since $\lim_{x\to 0} \frac{\sin 4x}{x} = 4 \lim_{x\to 0} \frac{\sin 4x}{4x} = 4(1) = 4$, the extended function is $y = \begin{cases} \frac{\sin 4x}{x}, & x \neq 0 \\ 4, & x = 0 \end{cases}.$

29. For $x \neq 4$ (and $x > 0$),

$$f(x) = \frac{x-4}{\sqrt{x}-2} = \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2} = \sqrt{x}+2.$$

The extended function is $y = \sqrt{x}+2$.

30. For $x \neq 2$ (and $x \neq -2$),

$$\begin{aligned} f(x) &= \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4} \\ &= \frac{(x-2)(x-5)(x+3)}{(x-2)(x+2)} \\ &= \frac{(x-5)(x+3)}{x+2} \\ &= \frac{x^2 - 2x - 15}{x+2}. \end{aligned}$$

The extended function is $y = \frac{x^2 - 2x - 15}{x+2}$.

31. The domain of f is all real numbers $x \neq 3$. f is continuous at all those points so f is a continuous function.

32. The domain of g is all real numbers $x > 1$. f is continuous at all those points so g is a continuous function.

33. f is the composite of two continuous functions $g \circ h$ where $g(x) = \sqrt{x}$ and $h(x) = \frac{x}{x+1}$.

34. f is the composite of two continuous functions $g \circ h$ where $g(x) = \sin x$ and $h(x) = x^2 + 1$.

35. f is the composite of three continuous functions $g \circ h \circ k$ where $g(x) = \cos x$, $h(x) = \sqrt[3]{x}$, and $k(x) = 1 - x$.

36. f is the composite of two continuous functions $g \circ h$ where $g(x) = \tan x$ and $h(x) = \frac{x^2}{x^2+4}$.

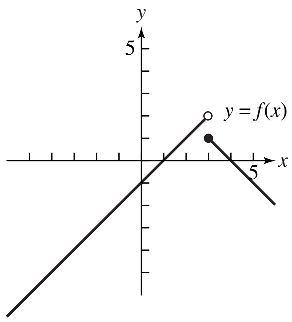
37. One possible answer:
Assume $y = x$, constant functions, and the square root function are continuous.
By the sum theorem, $y = x + 2$ is continuous.
By the composite theorem, $y = \sqrt{x+2}$ is continuous.
By the quotient theorem, $y = \frac{1}{\sqrt{x+2}}$ is continuous.
Domain: $(-2, \infty)$

38. One possible answer:
Assume $y = x$, constant functions, and the cube root function are continuous.
By the difference theorem, $y = 4 - x$ is continuous.
By the composite theorem, $y = \sqrt[3]{4-x}$ is continuous.
By the product theorem, $y = x^2 = x \cdot x$ is continuous.
By the sum theorem, $y = x^2 + \sqrt[3]{4-x}$ is continuous.
Domain: $(-\infty, \infty)$

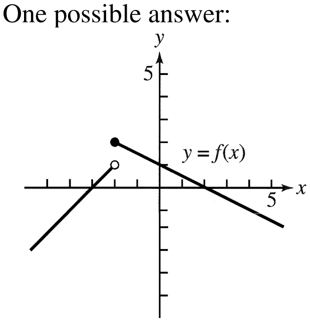
39. Possible answer:
Assume $y = x$ and $y = |x|$ are continuous.
By the product theorem, $y = x^2 = x \cdot x$ is continuous.
By the constant multiple theorem, $y = 4x$ is continuous.
By the difference theorem, $y = x^2 \ 4x$ is continuous.
By the composite theorem, $y = |x^2 - 4x|$ is continuous.
Domain: $(-\infty, \infty)$

40. One possible answer:
Assume $y = x$ and $y = 1$ are continuous.
Use the product, difference, and quotient theorems. One also needs to verify that the limit of this function as x approaches 1 is 2. Alternately, observe that the function is equivalent to $y = x + 1$ (for all x), which is continuous by the sum theorem.
Domain: $(-\infty, \infty)$

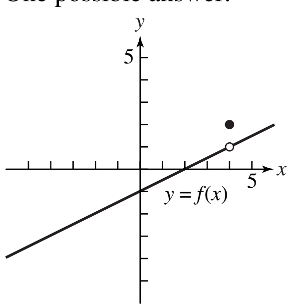
41. One possible answer:



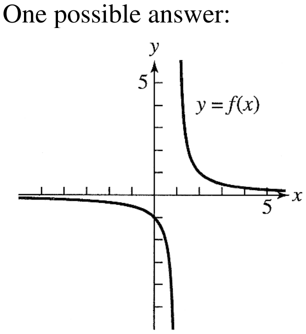
42. One possible answer:



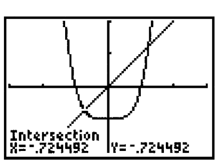
43. One possible answer:



44. One possible answer:



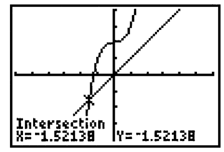
45.



[−3, 3] by [−2, 2]

Solving $x = x^4 - 1$, we obtain the solutions $x = -0.724$ and $x = 1.221$.

46.



[−6, 6] by [−4, 4]

Solving $x = x^3 + 2$, we obtain the solution $x = -1.521$.

47. Since $f(3) = \lim_{x\to 3^+} f(x) = 2a(3) = 6a$ and $\lim_{x\to 3^-} f(x) = 3^2 - 1 = 8$, the function will be continuous if $6a = 8$. Thus $a = \frac{4}{3}$.

48. Since $f(2) = \lim_{x\to 2^-} f(x) = 2(2) + 3 = 7$ and $\lim_{x\to 2^+} f(x) = 2a + 1$, the function will be continuous if $2a + 1 = 7$. Thus $a = 3$.

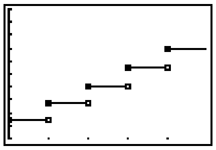
49. Since $f(-1) = \lim_{x\to -1^+} f(x) = a(-1)^2 - 1 = a - 1$ and $\lim_{x\to -1^-} f(x) = 4 - (-1)^2 = 3$, the function will be continuous if $a - 1 = 3$. Thus $a = 4$.

50. Since $f(1) = \lim_{x\to 1^+} f(x) = 1^3 = 1$ and $\lim_{x\to 1^-} f(x) = (1)^2 + 1 + a = 2 + a$, the function will be continuous if $2 + a = 1$. Thus $a = -1$.

51. Consider $f(x) = x - e^{-x}$. f is continuous, $f(0) = -1$, and $f(1) = 1 - \frac{1}{e} > 0.5$. By the Intermediate Value Theorem, for some c in $(0, 1)$, $f(c) = 0$ and $e^{-c} = c$.

52. **(a)** Luisa's salary is $\$36{,}500 = \$36{,}500\,(1.035)^0$ for the first year ($0 \le t < 1$), $\$36{,}500(1.035)$ for the second year ($1 \le t < 2$), $\$36{,}500\,(1.035)^2$ for the third year ($2 \le t < 3$), and so on. This corresponds to $y = 36{,}500\,(1.035)^{\operatorname{int} t}$.

(b)



[0, 4.8] by [35000, 45000]

The function is continuous at all points in the domain [0, 5) except at $t = 1, 2, 3, 4$.

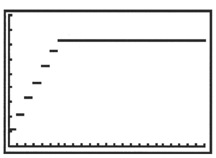
53. (a) We require:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1.10, & 0 < x \le 1 \\ 2.20, & 1 < x \le 2 \\ 3.30, & 2 < x \le 3 \\ 4.40, & 3 < x \le 4 \\ 5.50, & 4 < x \le 5 \\ 6.60, & 5 < x \le 6 \\ 7.25, & 6 < x \le 24. \end{cases}$$

This may be written more compactly as

$$f(x) = \begin{cases} -1.10 \ \text{int}(-x), & 0 \le x \le 6 \\ 7.25, & 6 < x \le 24 \end{cases}$$

(b)



[0, 24] by [0, 9]

This is continuous for all values of x in the domain [0, 24] except for $x = 0, 1, 2, 3, 4, 5, 6$.

54. False. Consider $f(x) = \frac{1}{x}$ which is continuous and has a point of discontinuity at $x = 0$.

55. False; if f has a jump discontinuity at $x = a$, then $\lim_{x \to a^-} f(x) \ne \lim_{x \to a^+} f(x)$, so f is not continuous at $x = a$.

56. B; $f(x) = \frac{1}{\sqrt{0}}$ is not defined.

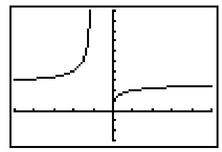
57. E; $f(x) = \sqrt{x-1} = \sqrt{1-1} = \sqrt{0}$ is the only defined option.

58. A; $f(1) = 1$.

59. E; $x = 3$ causes the denominator to be zero even after the rational expression is reduced.

60. (a) The function is defined when $1 + \frac{1}{x} > 0$, that is, on $(-\infty, -1) \cup (0, \infty)$. (It can be argued that the domain should also include certain values in the interval $(-1, 0)$, namely, those rational numbers that have odd denominators when expressed in lowest terms.)

(b)



[–5, 5] by [–3, 10]

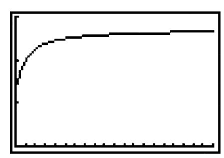
(c) If we attempt to evaluate $f(x)$ at these values, we obtain

$$f(-1) = \left(1 + \frac{1}{-1}\right)^{-1} = 0^{-1} = \frac{1}{0}$$

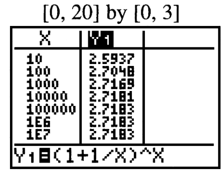
(undefined) and $f(0) = \left(1 + \frac{1}{0}\right)^0$ (undefined). Since f is undefined at these values due to division by zero, both values are points of discontinuity.

(d) The discontinuity at $x = 0$ is removable because the right-hand limit is 0. The discontinuity at $x = -1$ is not removable because it is an infinite discontinuity.

(e)



[0, 20] by [0, 3]



X	Y1
10	2.5937
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183
1E6	2.7183
1E7	2.7183

Y1=(1+1/X)^X

The limit is about 2.718, or e.

61. This is because $\lim_{h \to 0} f(a+h) = \lim_{x \to a} f(x)$.

62. Suppose not. Then f would be negative somewhere in the interval and positive somewhere else in the interval. So, by the Intermediate Value Theorem, it would have to be zero somewhere in the interval, which contradicts the hypothesis.

63. Since the absolute value function is continuous, this follows from the theorem about continuity of composite functions.

64. For any real number a, the limit of this function as x approaches a cannot exist. This is because as x approaches a, the values of the function will continually oscillate between 0 and 1.

Section 2.4 Rates of Change and Tangent Lines (pp. 87–95)

Quick Review 2.4

1. $\Delta x = 3-(-5) = 8$
$\Delta y = 5-2 = 3$

2. $\Delta x = a-1$
$\Delta y = b-3$

3. $m = \dfrac{-1-3}{5-(-2)} = \dfrac{-4}{7} = -\dfrac{4}{7}$

4. $m = \dfrac{3-(-1)}{3-(-3)} = \dfrac{4}{6} = \dfrac{2}{3}$

5. $y = \dfrac{3}{2}[x-(-2)]+3$
$y = \dfrac{3}{2}x+6$

6. $m = \dfrac{-1-6}{4-1} = \dfrac{-7}{3} = -\dfrac{7}{3}$
$y = -\dfrac{7}{3}(x-1)+6$
$y = -\dfrac{7}{3}x+\dfrac{25}{3}$

7. $y = -\dfrac{3}{4}(x-1)+4$
$y = -\dfrac{3}{4}x+\dfrac{19}{4}$

8. $m = -\dfrac{1}{-\frac{3}{4}} = \dfrac{4}{3}$
$y = \dfrac{4}{3}(x-1)+4$
$y = \dfrac{4}{3}x+\dfrac{8}{3}$

9. Since $2x + 3y = 5$ is equivalent to $y = -\dfrac{2}{3}x+\dfrac{5}{3}$, we use
$m = -\dfrac{2}{3}.$
$y = -\dfrac{2}{3}[x-(-1)]+3$
$y = -\dfrac{2}{3}x+\dfrac{7}{3}$

10. $\dfrac{b-3}{4-2} = \dfrac{5}{3}$
$b-3 = \dfrac{10}{3}$
$b = \dfrac{19}{3}$

Section 2.4 Exercises

1. (a) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(3)-f(2)}{3-2} = \dfrac{28-9}{1} = 19$

(b) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(1)-f(-1)}{1-(-1)} = \dfrac{2-0}{2} = 1$

2. (a) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(2)-f(0)}{2-0} = \dfrac{3-1}{2} = 1$

(b) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(12)-f(10)}{12-10} = \dfrac{7-\sqrt{41}}{2}$

3. (a) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(0)-f(-2)}{0-(-2)} = \dfrac{1-e^{-2}}{2}$

(b) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(3)-f(1)}{3-1} = \dfrac{e^3-e}{2}$

4. (a) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(4)-f(1)}{4-1} = \dfrac{\ln 4-0}{3} = \dfrac{\ln 4}{3}$

(b) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(103)-f(100)}{103-100}$
$= \dfrac{\ln 103-\ln 100}{3}$
$= \dfrac{1}{3}\ln\dfrac{103}{100}$
$= \dfrac{1}{3}\ln 1.03$

5. **(a)** $\frac{\Delta f}{\Delta x}=\frac{f\left(\frac{3\pi}{4}\right)-f\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}-\frac{\pi}{4}}=\frac{-1-1}{\frac{\pi}{2}}=-\frac{4}{\pi}$

(b) $\frac{\Delta f}{\Delta x}=\frac{f\left(\frac{\pi}{2}\right)-f\left(\frac{\pi}{6}\right)}{\frac{\pi}{2}-\frac{\pi}{6}}=\frac{0-\sqrt{3}}{\frac{\pi}{3}}=-\frac{3\sqrt{3}}{\pi}$

6. **(a)** $\frac{\Delta f}{\Delta x}=\frac{f(\pi)-f(0)}{\pi-0}=\frac{1-3}{\pi}=-\frac{2}{\pi}$

(b) $\frac{\Delta f}{\Delta x}=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}=\frac{1-1}{2\pi}=0$

7. We use $Q_1=(10, 225)$, $Q_2=(14, 375)$, $Q_3=(16.5, 475)$, $Q_4=(18, 550)$, and $P=(20, 650)$.

(a) Slope of PQ_1: $\frac{650-225}{20-10}\approx 43$

Slope of PQ_2: $\frac{650-375}{20-14}\approx 46$

Slope of PQ_3: $\frac{650-475}{20-16.5}=50$

Slope of PQ_4: $\frac{650-550}{20-18}=50$

Secant	Slope
PQ_1	43
PQ_2	46
PQ_3	50
PQ_4	50

The appropriate units are meters per second.

(b) Approximately 50 m/sec

8. We use $Q_1=(5, 20)$, $Q_2=(7, 38)$, $Q_3=(8.5, 56)$, $Q_4=(9.5, 72)$, and $P=(10, 80)$.

(a) Slope of PQ_1: $\frac{80-20}{10-5}=12$

Slope of PQ_2: $\frac{80-38}{10-7}=14$

Slope of PQ_3: $\frac{80-56}{10-8.5}=16$

Slope of PQ_4: $\frac{80-72}{10-9.5}=16$

Secant	Slope
PQ_1	12
PQ_2	14
PQ_3	16
PQ_4	16

The appropriate units are meters per second.

(b) Approximately 16 m/sec

9. **(a)** $\lim_{h\to 0}\frac{y(-2+h)-y(-2)}{h}$

$=\lim_{h\to 0}\frac{(-2+h)^2-(-2)^2}{h}$

$=\lim_{h\to 0}\frac{4-4h+h^2-4}{h}$

$=\lim_{h\to 0}\frac{-4h+h^2}{h}$

$=\lim_{h\to 0}(-4+h)$

$=-4$

(b) The tangent line has slope –4 and passes through (–2, y(–2)) = (–2, 4)

$y=-4[x-(-2)]+4$

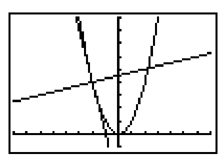
$y=-4x-4$

(c) The normal line has slope $-\frac{1}{-4}=\frac{1}{4}$ and passes through (–2, y(–2)) = (–2, 4).

$y=\frac{1}{4}[x-(-2)]+4$

$y=\frac{1}{4}x+\frac{9}{2}$

(d)



[–8, 7] by [–1, 9]

10. **(a)** $\lim_{h\to 0}\frac{y(1+h)-y(1)}{h}$

$= \lim_{h\to 0}\frac{[(1+h)^2-4(1+h)]-[1^2-4(1)]}{h}$

$= \lim_{h\to 0}\frac{1+2h+h^2-4-4h+3}{h}$

$= \lim_{h\to 0}\frac{h^2-2h}{h}$

$= \lim_{h\to 0}(h-2)$

$= -2$

(b) The tangent line has slope –2 and passes through (1, y(1)) = (1, –3).

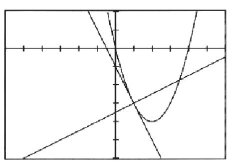
$y = -2(x-1)-3$

$y = -2x-1$

(c) The normal line has slope $-\frac{1}{-2}=\frac{1}{2}$ and passes through (1, y(1)) = (1, –3).

$y = \frac{1}{2}(x-1)-3$

$y = \frac{1}{2}x-\frac{7}{2}$

(d) 

[–6, 6] by [–6, 2]

11. **(a)** $\lim_{h\to 0}\frac{y(2+h)-y(2)}{h} = \lim_{h\to 0}\frac{\frac{1}{(2+h)-1}-\frac{1}{2-1}}{h}$

$= \lim_{h\to 0}\frac{\frac{1}{h+1}-1}{h}$

$= \lim_{h\to 0}\frac{1-(h+1)}{h(h+1)}$

$= \lim_{h\to 0}\left(-\frac{1}{h+1}\right)$

$= -1$

(b) The tangent line has slope −1 and passes through (2, y(2)) = (2, 1).

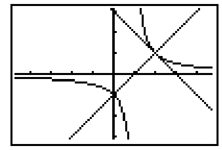
$y = -(x-2)+1$

$y = -x+3$

(c) The normal line has slope $-\frac{1}{-1}=1$ and passes through (2, y(2)) = (2, 1).

$y = 1(x-2)+1$

$y = x-1$

(d) 

[–4.7, 4.7] by [–3.1, 3.1]

12. **(a)** $\lim_{h\to 0}\frac{y(0+h)-y(0)}{h}$

$= \lim_{h\to 0}\frac{(h^2-3h-1)-(-1)}{h}$

$= \lim_{h\to 0}\frac{h^2-3h}{h}$

$= \lim_{h\to 0}(h-3)$

$= -3$

(b) The tangent line has slope –3 and passes through (0, y(0)) = (0, –1).

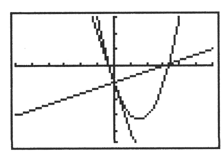
$y = -3(x-0)-1$

$y = -3x-1$

(c) The normal line has slope $-\frac{1}{-3}=\frac{1}{3}$ and passes through (0, y(0)) = (0, –1).

$y = \frac{1}{3}(x-0)-1$

$y = \frac{1}{3}x-1$

(d) 

[–6, 6] by [–5, 3]

13. **(a)** Near $x = 2, f(x) = |x| = x$.

$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(2+h)-2}{h}$

$= \lim_{h\to 0} 1$

$= 1$

(b) Near $x = -3, f(x) = |x| = -x$.

$\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} = \lim_{h\to 0}\frac{(3-h)-3}{h}$

$= \lim_{h\to 0} -1$

$= -1$

14. Near $x = 1$, $f(x) = |x - 2| = -(x - 2) = 2 - x$.

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{[2-(1+h)]-(2-1)}{h}\\ &= \lim_{h\to 0}\frac{1-h-1}{h}\\ &= \lim_{h\to 0} -1\\ &= -1\end{aligned}$$

15. First, note that $f(0) = 2$.

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-}\frac{(2-2h-h^2)-2}{h}\\ &= \lim_{h\to 0^-}\frac{-2h-h^2}{h}\\ &= \lim_{h\to 0^-}(-2-h)\\ &= -2\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{(2h+2)-2}{h}\\ &= \lim_{h\to 0^+} 2\\ &= 2\end{aligned}$$

No, the slope from the left is –2 and the slope from the right is 2. The two-sided limit of the difference quotient does not exist.

16. First, note that $f(0) = 0$.

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h-0}{h} = -1$$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{(h^2-h)-0}{h}\\ &= \lim_{h\to 0^+}(h-1)\\ &= -1\end{aligned}$$

Yes. The slope is -1.

17. First, note that $f(2) = \frac{1}{2}$

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-}\frac{\frac{1}{2+6}-\frac{1}{2}}{h}\\ &= \lim_{h\to 0^-}\frac{2-(2+h)}{2h(2+h)}\\ &= \lim_{h\to 0^-}\frac{-h}{2h(2+h)}\\ &= \lim_{h\to 0^-} -\frac{1}{2(2+h)}\\ &= -\frac{1}{4}\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+}\frac{\frac{4-(2+h)}{4}-\frac{1}{2}}{h}\\ &= \lim_{h\to 0^+}\frac{[4-(2+h)]-2}{4h}\\ &= \lim_{h\to 0^+}\frac{-h}{4h}\\ &= -\frac{1}{4}\end{aligned}$$

Yes. The slope is $-\frac{1}{4}$.

18. No; the function is discontinuous at $x = \frac{3\pi}{4}$ because

$$\lim_{x\to(3\pi/4)^-} f(x) = \lim_{x\to(3\pi/4)^-} \sin x = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

but $f\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$.

19. (a)

$$\begin{aligned}&\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\ &= \lim_{h\to 0}\frac{[(a+h)^2+2]-(a^2+2)}{h}\\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2+2-a^2-2}{h}\\ &= \lim_{h\to 0}\frac{2ah+h^2}{h}\\ &= \lim_{h\to 0}(2a+h)\\ &= 2a\end{aligned}$$

(b) The slope of the tangent steadily increases as a increases.

20. (a)

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\frac{2}{a+6}-\frac{2}{a}}{h}\\ &= \lim_{h\to 0}\frac{2a-2(a+h)}{ah(a+h)}\\ &= \lim_{h\to 0}\frac{-2}{a(a+h)}\\ &= -\frac{2}{a^2}\end{aligned}$$

(b) The slope of the tangent is always negative. The tangents are very steep near $x = 0$ and nearly horizontal as a moves away from the origin.